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MOMENTLESS MODEL OF THE ELASTOPLASTIC DEFORMATION AND LIMITING STATE OF THIN INTERLAYERS

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There have been many investigations of the two-dimensional plastic deformation of layers positioned between rigid blocks [1, 2], but the studies have been restricted to examination of the limiting state of straight layers of uniform thickness subjected to special types of loads. Below, we formulate a plane-strain model for interlayers with the goal of modeling the entire process of elastoplastic deformation of the layers – from the moment plastic strains develop to attainment of the limiting state. The layers are will be subjected to a combination of tensile (compressive), shear, and bending loads. We will consider the general case, when a layer may be curvilinear and of variable thickness.

1. Approximation of the Strain-Rate Dependence of Stress. We use the following equations [3] to describe elastoplastic deformation and establish the dependence of the stresses on the mean strain rates within the time interval [t, t + τ]

$$\begin{gathered}\sigma'_{ij}(t+\tau) = \sigma'_{ij}(t) + \tau[2\mu e'_{ij} - \lambda\sigma'_{ij}(t+\tau)],\\ \sigma(t+\tau) = \sigma(t) + K\tau e,\ K = E/[3(1-2\nu)],\\ e'_{ij} = e_{ij} - \frac{1}{3}\delta_{ij}e,\ \sigma'_{ij} = \sigma_{ij} - \delta_{ij}\sigma,\ e = \delta^{ij}e_{ij},\ \sigma = \frac{1}{3}\delta^{ij}\sigma_{ij},\end{gathered} \tag{1.1}$$

where σ_{ij} and e_{ij} are components of the stress and strain-rate tensors in a cartesian coordinate system; λ is a non-negative quantity; $\lambda = 0$ at $J_2(t + \tau) < J_2^*$. The value of λ is determined by the condition $J_2(t + \tau) = J_2^*$ in the case of ideal plasticity and the condition

$$\lambda = \frac{(1-\varkappa)[J_2(t+\tau) - J_2^*]}{2\varkappa J_2(t+\tau)}$$

in the case of isotropic strain-hardening. Here, $J_2(t + \tau) = 1/2\sigma_{ij}'(t + \tau)\sigma_{ij}'(t + \tau)$; $J_2^* = \max(J_2^s, \max J_2)$; J_2^s is the value of J_2 at which an element of the medium first begins to deform plastically; $\max J_2$ is the largest value of J_2 for the entire history of deformation of the element; $\varkappa = \mu'/\mu$ is the strain-hardening coefficient; μ' is the shear modulus on the curve describing pure shear of the element; μ is the shear modulus; E is the elastic modulus; ν is the Poisson's ratio.

The method used in [4] to prove the uniqueness of stress intensities in problems concerning elastoplastic deformation can be used here to prove that the problem of determining the stresses $\sigma_{ij}(t + \tau)$ so as to satisfy (1.1), the equilibrium equations

$$\frac{\partial\sigma_{ij}(t+\tau)}{\partial x_j} + f_i = 0 \tag{1.2}$$

and the boundary conditions

$$\sigma_{ij}(t+\tau)n_j|_{S_\sigma} = p_i^0,\ u_i|_{S_u} = u_i^0,$$

can have only a unique solution. In the case of isotropic strain-hardening, the problem of finding the strain rates e_{ij} will also have a unique solution. The value of λ may be ambiguously determined by Eqs. (1.1) in the case of ideal plasticity. Here, $\sigma_{ij}(t + \tau)$ may correspond to values of e_{ij}' besides those found by the algorithm that we will present here.

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Due to the nonlinearity of Eqs. (1.1), they can be satisfied only by successive approximation. Below, we use a process of successive approximation that consists of two alternating stages: 1) solution of the problem of the deformation of an interface within the interval $[t, t + \tau]$ with the dependence of λ on the coordinates assigned in (1.1); 2) calculation of λ from strain rates found at the previous stage by means of the following equations [3]:

in the case of ideal plasticity

$$\tau\lambda = 0 \quad \text{when} \quad J_2^e \leq J_2^*, \quad \tau\lambda = \sqrt{J_2^e / J_2^*} - 1 \quad \text{when} \quad J_2^e > J_2^*,$$

in the case of isotropic strain-hardening

$$\frac{1}{1 + \tau\lambda} = \frac{\varkappa + \sqrt{\varkappa^2 + (1 - \varkappa^2)(J_2^* / J_2^e)}}{1 + \varkappa}.$$

Here

$$J_2^e = \frac{1}{2}\sigma_{ij}''^{e}\sigma_{ij}''^{e}; \quad \sigma_{ij}''^{e} = 2\mu\tau e_{ij}' + \sigma_{ij}'(t).$$

In the first approximation, we assume that $\lambda = 0$ everywhere in the interface.

2. Stiffness Equations of Elements of the Interface. In the momentless model being discussed here to describe the plane strain of an interface, we represent the latter as consisting of tetragonal elements (Fig. 1a, b). By the stiffness equations of an element, we mean the dependence of the forces at the boundaries of the element on the mean velocities of the boundaries. In constructing these equations for each element, we introduce an oblique coordinate system $\xi^1, \xi^2 \in [-1, 1]$ [5]. Equations (1.1) are written in the form

$$\begin{gathered}
\hat{\sigma}^{\alpha\beta}(t + \tau) = a^{\alpha\beta ij}\tau \hat{e}_{ij} + \hat{\sigma}_{\varkappa}^{\alpha\beta}, \quad \hat{e}_{\alpha\beta} = \frac{1}{2}\left(\frac{\partial \mathbf{u}}{\partial \xi^\alpha} \cdot \hat{\mathbf{Э}}_\beta + \frac{\partial \mathbf{u}}{\partial \xi^\beta} \cdot \hat{\mathbf{Э}}_\alpha\right), \\
\hat{\sigma}_{\varkappa}^{\alpha\beta} = (1 - \tilde{\lambda})\hat{g}^{\alpha\beta}\sigma(t) + \tilde{\lambda}\hat{\sigma}^{\alpha\beta}(t), \quad \sigma(t) = \frac{1}{3}\,[\hat{g}_{\alpha\beta}\hat{\sigma}^{\alpha\beta}(t) + \sigma^{33}(t)], \\
\sigma^{33}(t + \tau) = \left(K - \frac{2}{3}\mu\tilde{\lambda}\right)\hat{g}^{\alpha\beta}\tau \hat{e}_{\alpha\beta} + (1 - \tilde{\lambda})\sigma(t) + \tilde{\lambda}\sigma^{33}(t), \\
a^{\alpha\beta ij} = \left(K - \frac{2}{3}\mu\tilde{\lambda}\right)\hat{g}^{\alpha\beta}\hat{g}^{ij} + \mu\tilde{\lambda}(\hat{g}^{\alpha i}\hat{g}^{\beta j} + \hat{g}^{\alpha j}\hat{g}^{\beta i}), \\
\hat{g}^{\alpha\beta} = \hat{Э}^\alpha \cdot \hat{Э}^\beta, \quad \hat{Э}^\alpha = \frac{\partial \xi^\alpha}{\partial x^i}\mathbf{e}^i, \quad \hat{Э}_\alpha = \frac{\partial x^i}{\partial \xi^\alpha}\mathbf{e}_i, \quad \alpha,\beta = 1,2, \quad \tilde{\lambda} = \frac{1}{(1 + \tau\lambda)},
\end{gathered} \tag{2.1}$$

while equations (1.2) are written in the form

$$\begin{gathered}
\frac{\partial \mathbf{p}^\alpha(t + \tau)}{\partial \xi^\alpha} + \sqrt{g}\mathbf{f} = 0, \quad \mathbf{p}^\alpha(t + \tau) = A^{\alpha\beta}\tau\frac{\partial \mathbf{u}}{\partial \xi^\beta} + \mathbf{p}_{\varkappa}^\alpha, \\
A^{\alpha\beta} = \sqrt{g}a^{\alpha ij\beta}\hat{Э}_i\hat{Э}_j, \quad \mathbf{p}_{\varkappa}^\alpha = \hat{\sigma}_{\varkappa}^{\alpha\beta}\sqrt{g}\hat{Э}_\beta, \quad \sqrt{g} = |Э_1 \times Э_2|,
\end{gathered} \tag{2.2}$$

where $\hat{\sigma}^{\alpha\beta}(t + \tau)$ and $\hat{e}_{\alpha\beta}$ are components of the stress and strain-rate tensors in the coordinate system ξ^α; $\mathbf{e}_i$ are the basis vectors of the cartesian coordinate system x^i.

Below, we simplify the notation in the formulas by designating $\hat{\sigma}^{\alpha\beta}(t + \tau)$, $\mathbf{p}^\alpha(t + \tau)$ as $\hat{\sigma}^{\alpha\beta}$, $\mathbf{p}^\alpha$.

In constructing the stiffness equations, we make use of an approximation for the vectors $\mathbf{p}^\alpha$

$$\mathbf{p}^\alpha = \overset{(0)}{\mathbf{p}^\alpha} + \overset{(1)}{\mathbf{p}^\alpha}\xi^\alpha \quad (\alpha = 1,2) \tag{2.3}$$

and three approximations for the velocity vector:

$$\overset{(0)}{\mathbf{u}}, \quad \mathbf{u}^\alpha = \overset{(0)}{\mathbf{u}^\alpha} + \overset{(1)}{\mathbf{u}^\alpha}\xi^\alpha \quad (\alpha = 1,2). \tag{2.4}$$

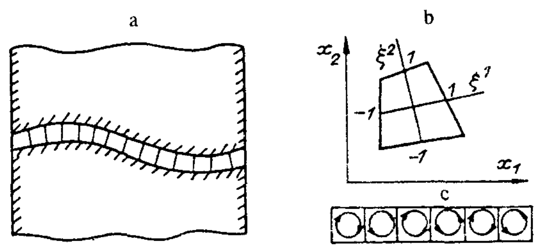

Fig. 1

Here, $\overset{(0)}{\mathbf{u}}$ is the mean velocity in the given element; $\mathbf{u}^\alpha$ represents the mean velocities on the lines ξ^α = const. We assume that $\mathbf{p}^\alpha$, $\overset{(0)}{\mathbf{u}}$, and $\mathbf{u}^\alpha$ are connected by the equations

$$\int\limits_\omega \left(\frac{\partial \mathbf{p}^\alpha}{\partial \xi^\alpha} + \sqrt{g}\mathbf{f}\right) d\omega = 0, \ \int\limits_\omega \left(\mathbf{p}^\alpha - A^{\alpha\beta}\tau\frac{\partial \mathbf{u}^\beta}{\partial \xi^\beta} - \mathbf{p}^\alpha_x - \mathbf{q}^\alpha\right) d\omega = 0,$$
$$\mathbf{q}^1 = \gamma_1\sqrt{g}a^{1212}\tau\left(\frac{\partial \mathbf{u}^1}{\partial \xi^1}\cdot\hat{\ni}_2\right)\hat{\ni}_2, \ \mathbf{q}^2 = \gamma_2\sqrt{g}a^{1212}\tau\left(\frac{\partial \mathbf{u}^2}{\partial \xi^2}\cdot\hat{\ni}_1\right)\hat{\ni}_1, \tag{2.5}$$
$$\overset{(0)}{\mathbf{u}^\alpha} - \overset{(0)}{\mathbf{u}} - \Lambda^\alpha\frac{\partial \mathbf{p}^\alpha}{\partial \xi^\alpha} = 0, \ \omega = \{\xi^1, \xi^2 \in [-1,1]\},$$

where γ_1 and γ_2 are positive constants determining the forces associated with viscous drag $\mathbf{q}^\alpha$. These forces are introduced into Eqs. (2.5) to suppress parasitic rotations of the type indicated in Fig. 1c. For $\mathbf{q}^\alpha = 0$, such rotations can arise in elements when the boundary conditions on the surfaces of the interface are formulated in the form of conditions for shear stresses. Past numerical calculations have shown that assigning $\gamma_1 = \gamma_2 = 0.05$ is usually sufficient to suppress parasitic rotations.

In (2.5), Λ^α is a tensor satisfying the condition $\Lambda^\alpha \mathbf{a}\cdot\mathbf{a} \geq 0$. An equality is possible only if $\mathbf{a} = 0$. In the model being discussed, we take

$$\Lambda^\alpha = (B^\alpha)^{-1}, \ B^\alpha = \frac{3}{4}\tau\int\limits_\omega A^{\alpha\alpha} d\omega. \tag{2.6}$$

With such a choice of Λ^α, stiffness equations of the element based on linear approximations (2.3)-(2.4) are close to the stiffness equations based on quadratic approximations analogous to those used in [6, 7]. In the case of rectangular elastic elements, Eqs. (2.3)-(2.6) are analogous to the equations employed in [8, 9] to construct the stiffness equations of elastic elements.

We designate

$$\mathbf{p}^\alpha|_{\xi^\alpha = \pm 1} = \mathbf{p}^\alpha_\pm, \ \mathbf{u}^\alpha|_{\xi^\alpha = \pm 1} = \mathbf{u}^\alpha_\pm. \tag{2.7}$$

It follows from (2.3)-(2.7) that

$$\mathbf{p}^1_+ - \mathbf{p}^1_- + \mathbf{p}^2_+ - \mathbf{p}^2_- + 2(\sqrt{g}\mathbf{f})^{(0)} = 0,$$
$$\mathbf{p}^\alpha_+ - \mathbf{p}^\alpha_- - 3(A^{\alpha\alpha})^{(0)}\tau(\mathbf{u}^\alpha_+ + \mathbf{u}^\alpha_- - 2\overset{(0)}{\mathbf{u}}) = 0, \tag{2.8}$$
$$\mathbf{p}^\alpha_+ + \mathbf{p}^\alpha_- - (A^{\alpha\beta})^{(0)}\tau(\mathbf{u}^\beta_+ - \mathbf{u}^\beta_-) - 2(\mathbf{p}^\alpha_x + \mathbf{q}^\alpha)^{(0)} = 0,$$

where the symbol $(\ )^{(0)}$ denotes averaging over ω. Excluding $\overset{(0)}{\mathbf{u}}$ from (2.8), we find that the stiffness equations of an element of the interface can be written in the form

$$\mathbf{p}^\alpha_+ - \mathbf{p}^\alpha_- = D^{\alpha\beta}\tau(\mathbf{u}^\beta_+ + \mathbf{u}^\beta_-) + 2\chi^\alpha,$$
$$\mathbf{p}^\alpha_+ + \mathbf{p}^\alpha_- = C^{\alpha\beta}\tau(\mathbf{u}^\beta_+ - \mathbf{u}^\beta_-) + 2(\mathbf{p}^\alpha_x)^{(0)}, \tag{2.9}$$

where $D^{\alpha\beta}$, $C^{\alpha\beta}$, and χ^α depend only on the coefficients of Eqs. (2.8).

It follows from (2.3)-(2.5) that

$$\int_{\omega}\left[\frac{\partial}{\partial\xi^{\alpha}}(\mathbf{p}^{\alpha}\cdot\mathbf{u}^{\alpha})+\mathbf{f}\cdot\overset{(0)}{\mathbf{u}}\right]d\omega=\int_{\omega}\Big\{\Big(A^{\alpha\beta}\tau\frac{\partial\mathbf{u}^{\beta}}{\partial\xi^{\beta}}\Big)\cdot\frac{\partial\mathbf{u}^{\alpha}}{\partial\xi^{\alpha}}+\Big(\Lambda^{\alpha}\frac{\partial\mathbf{p}^{\alpha}}{\partial\xi^{\alpha}}\Big)\cdot\frac{\partial\mathbf{p}^{\alpha}}{\partial\xi^{\alpha}}$$
$$+\sqrt{g}a^{1212}\tau\left[\gamma_1\left(\frac{\partial\mathbf{u}^1}{\partial\xi^1}\cdot\dot{\mathbf{Э}}_2\right)^2+\gamma_2\left(\frac{\partial\mathbf{u}^2}{\partial\xi^2}\cdot\dot{\mathbf{Э}}_1\right)^2\right]+\mathbf{p}^{\alpha}_{\times}\cdot\frac{\partial\mathbf{u}^{\alpha}}{\partial\xi^{\alpha}}\Big\}d\omega. \tag{2.10}$$

Using (2.10), we can show that, with assigned $\mathbf{p}^{\alpha}_{\pm}$, Eqs. (2.9) determine the velocities $\mathbf{u}^{\alpha}_{\pm}$ to within the rates of translation of the element.

We use $\mathbf{u}_*$ and $\mathbf{p}_*^{\alpha}$ to represent the solution of Eqs. (2.1)-(2.2). It follows from (2.1)-(2.5) that

$$\int_{\omega}\Big\{A^{\alpha\beta}\tau\frac{\partial(\mathbf{u}_*-\mathbf{u}^{\beta})}{\partial\xi^{\beta}}\cdot\frac{\partial(\mathbf{u}_*-\mathbf{u}^{\alpha})}{\partial\xi^{\alpha}}+\Lambda^{\alpha}\frac{\partial\mathbf{p}^{\alpha}}{\partial\xi^{\alpha}}\cdot\frac{\partial\mathbf{p}^{\alpha}}{\partial\xi^{\alpha}}+\mathbf{q}^{\alpha}\cdot\frac{\partial\mathbf{u}^{\alpha}}{\partial\xi^{\alpha}}\Big\}d\omega$$
$$=\int_{\omega}\Big\{\frac{\partial}{\partial\xi^{\alpha}}[(\mathbf{p}^{\alpha}_*-\mathbf{p}^{\alpha})\cdot(\mathbf{u}_*-\mathbf{u}^{\alpha})]+[\sqrt{g}\mathbf{f}-(\sqrt{g}\mathbf{f})^{(0)}]\cdot\mathbf{u}_*$$
$$+\Big(\mathbf{u}^{\alpha}-\overset{(0)}{\mathbf{u}^{\alpha}}-\Lambda^{\alpha}\frac{\partial\mathbf{p}^{\alpha}}{\partial\xi^{\alpha}}\Big)\cdot\frac{\partial\mathbf{p}^{\alpha}_*}{\partial\xi^{\alpha}}+[(A^{\alpha\beta})^{(0)}-A^{\alpha\beta}]\tau\frac{\partial\mathbf{u}^{\beta}}{\partial\xi^{\beta}}\cdot\frac{\partial\mathbf{u}_*}{\partial\xi^{\alpha}} \tag{2.11}$$
$$+[(\mathbf{p}^{\alpha}_{\times})^{(0)}-\mathbf{p}^{\alpha}_{\times}+\mathbf{p}^{\alpha}-\overset{(0)}{\mathbf{p}^{\alpha}}+(\mathbf{q}^{\alpha})^{(0)}]\cdot\frac{\partial\mathbf{u}_*}{\partial\xi^{\alpha}}\Big\}d\omega.$$

Considering that $\mathbf{p}_*^{\alpha}$ is a quantity on the order of the product of the stress vector and the linear dimension of the element, we find that the right side of (2.11) consists of terms on the order of this dimension and terms on the order of γ_1, γ_2. It follows from this that by representing the interface as consisting of a successively increasing number of layers of elements with (2.9) as their stiffness equations, we can obtain a sequence of solutions that converges to the solution of Eqs. (2.1)-(2.2). The present model of interface deformation, involving representation of the layer in the form of a single layer of elements, is the first approximation in this sequence of solutions.

3. Conditions on the Surface of the Interface. We will restrict ourselves to the class of problem in which the displacements of the blocks are assigned as functions of time. We use $\mathbf{w}_{\pm}$ to represent the velocities of particles of the rigid blocks at their interfaces with the interface.

In the model we are considering, normal components of velocities at interfaces assumed to be continuous:

$$(\mathbf{u}^2_{\pm}-\mathbf{w}_{\pm})\cdot\dot{\mathbf{Э}}^2|_{\xi^2=\pm1}=0. \tag{3.1}$$

The shear stresses on the surfaces of the interface cannot exceed the value $\tau_*=\sqrt{J_2^*}$. We therefore take the following in the given model

$$(\mathbf{u}^2_+-\mathbf{w}_+)\cdot\mathbf{Э}_1|_{\xi^2=1}=0 \tag{3.2}$$

with

$$|\mathbf{p}^2_+\cdot\dot{\mathbf{Э}}_1|\big|_{\xi^2=1}\leq\tau_*|\dot{\mathbf{Э}}_1|^2\big|_{\xi^2=1}. \tag{3.3}$$

If inequality (3.3) is violated under condition (3.2), we replace (3.2) by the equality

$$(\mathbf{p}^2_+\cdot\dot{\mathbf{Э}}_1)\big|_{\xi^2=1}=\pm\tau_*|\dot{\mathbf{Э}}_1|^2\big|_{\xi^2=1}. \tag{3.4}$$

Here, we assume that the sign in the right side of (3.4) is the same as the sign of $(\mathbf{p}_+^2\cdot\dot{\mathbf{Э}}_1)|_{\xi^2=1}$ under condition (3.2). If the following inequality is violated under condition (3.4)

$$[(\mathbf{p}^2_+\cdot\dot{\mathbf{Э}}_1)(\mathbf{w}_+-\mathbf{u}^2_+)\cdot\dot{\mathbf{Э}}_1]\big|_{\xi^2=1}\geq0, \tag{3.5}$$

(3.4) is replaced by condition (3.2). Inequality (3.5) means the rate of dissipation is negative during slip of the layer.

We formulate the conditions for the tangential components of the velocities and the stress vectors on the surfaces $\xi^2=-1$ of elements of the interface by analogy with (3.2)-(3.5).

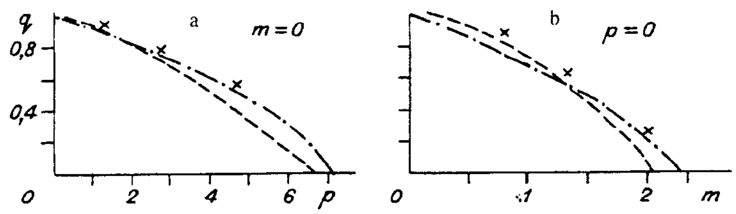

Fig. 2

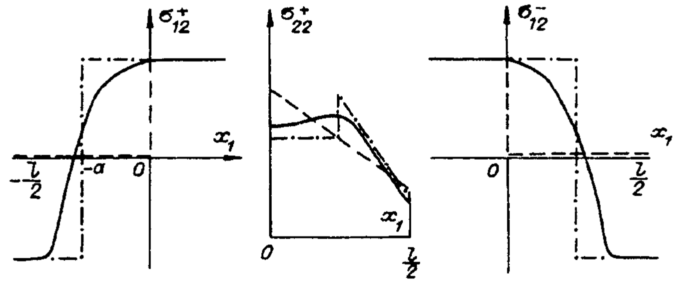

Fig. 3

4. Algebraic Equations of the Momentless Model of the Interface. Proceeding on the basis of (2.9) and boundary conditions (3.1)-(3.5) on the boundaries $\xi^2 = \pm 1$ of elements of the interface, we find that

$$\mathbf{p}_+^1 = A\mathbf{u}_+^1 + B\mathbf{u}_-^1 + \varphi,\ \mathbf{p}_-^1 = C\mathbf{u}_+^1 + D\mathbf{u}_-^1 + \psi. \tag{4.1}$$

Equations (4.1) and the conditions expressing continuity of the vectors $\mathbf{u}^1$ $\mathbf{p}^1$ on the interfaces of adjacent elements form the system of equations

$$\begin{gathered}\mathbf{p}_{i+1}^1 = \tau(A_{i+1/2}\mathbf{u}_{i+1}^1 + B_{i+1/2}\mathbf{u}_i^1) + \varphi_{i+1/2},\\ \mathbf{p}_i^1 = \tau(C_{i+1/2}\mathbf{u}_{i+1}^1 + D_{i+1/2}\mathbf{u}_i^1) + \psi_{i+1/2},\\ i = 0,\ 1,\ \dots,\ N-1,\end{gathered} \tag{4.2}$$

where N is the number of elements. If we add the following boundary conditions to (4.2)

$$\mathbf{p}_0^1 = L_0\tau\mathbf{u}_0^1 + \varphi_0,\ \mathbf{p}_N^1 = L_N\tau\mathbf{u}_N^1 + \psi_N, \tag{4.3}$$

we obtain a system of algebraic equations that is closed relative to $\mathbf{u}_i^1$, $\mathbf{p}_i^1$($i = 0, 1,\dots, N$). This system represents the model of the interface. The coefficients of system (4.2)-(4.3) depend on τ and λ and the form of the conditions for the tangential components of the velocities and stress vectors on the boundaries $\xi^2 = \pm 1$ of the elements.

With assigned τ and λ and the above conditions on boundaries $\xi^2 = \pm 1$, we can solve system (4.2)-(4.3) by the trial-run method. We use the solution that is obtained to correct the value of λ as indicated in Part 1. We also correct the boundary conditions on $\xi^2 = \pm 1$ as described in Part 3. The iteration is complete when conditions (3.2)-(3.5) and the analogous conditions on the boundaries $\xi^2 = -1$ of the elements are satisfied and the values of λ in the next iteration differ little from λ in the previous iteration.

5. Limiting State of a Straight Layer. To illustrate the use of the model formulated above to describe interface deformation, we examined the ideal elastoplastic deformation of a straight interface in the case when one block was stationary and the other underwent translation and rotation. We calculated the forces P and Q and moment M necessary for deformation:

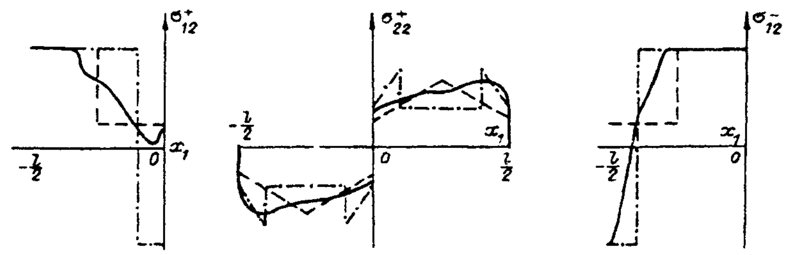

Fig. 4

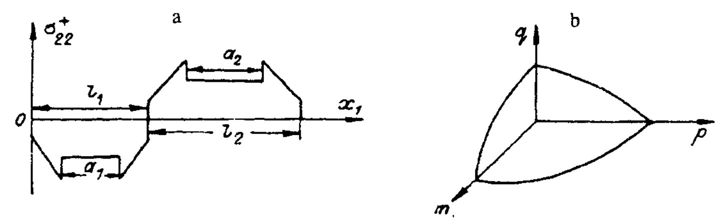

Fig. 5

$$P = \int_{-l/2}^{l/2} \sigma_{22}^{+} dx_1,\ Q = \int_{-l/2}^{l/2} \sigma_{12}^{+} dx_1,\ M^{\pm} = \int_{-l/2}^{l/2} \sigma_{22}^{\pm} x_1 dx_1,$$

$$M = M^{+} - \frac{1}{2}Qh = M^{-} + \frac{1}{2}Qh = \frac{1}{2}(M^{+} + M^{-}).$$

Here, l and h are the length and thickness of the interlayer; $\sigma_{22}^{\pm}$ and $\sigma_{12}^{\pm}$ are the normal and shear stresses on the boundaries $\xi^2 = \pm 1$ of the elements of the interface. The calculations were continued until we reached the limiting state, i.e. the state in which the interface was deformed with constant P, Q, and M. The circles in Fig. 2 show the limit loads in tension with shear and in bending with shear for an interface with l/h = 10.

$$q = \frac{Q}{\tau_* l},\ p = \frac{P}{\tau_* l},\ m = \frac{2M}{\tau_* l^2}.$$

The dashed lines in Fig. 2 show the limit loads corresponding to the approximate solutions obtained by L. M. Kachanov in [1, 10].

Figure 3 shows the distribution of the normal and shear stresses on the surfaces of an interface in the limiting state in tension with shear. The solid lines show the numerical solution obtained with q = 0.51 and p = 5.03, while the dashed lines show Kachanov's solution for q = 0.51. Since the stresses $\sigma_{22}^{\pm}$ are symmetric relative to $x_1 = 0$, in Fig. 3 we show only their distribution at $x_1 \geq 0$.

Figure 4 shows the distribution of the normal and shear stresses on the surfaces of the interface in the limiting state in bending with shear. The solid lines show the numerical solution obtained with q = 0.62 and m = 1.42, while the dashed lines show Kachanov's solution [10] for q = 0.62. Since the graphs of the shear stresses σ_{12}^{+}, σ_{12}^{-} are symmetric relative to $x_1 = 0$, in Fig. 4 we show only their distribution at $x_1 \leq 0$.

The dot-dash lines in Figs. 2a and 3 correspond to the approximate solution obtained from the following equations for the problem of the limiting state of an interface in tension with shear

$$\sigma_{12}^{\pm} = \mp \tau_*,\ h\frac{\partial \sigma_{11}}{\partial x_1} = 2\tau_*,\ \sigma_{11}\Big|_{x_1 = -l/2} = 0,\ \sigma_{22} - \sigma_{11} = 2\tau_* \quad \text{when } -l/2 \leq x_1 \leq -a; \tag{5.1}$$

$$\sigma_{12}^{+} = \sigma_{12}^{-} = \tau_*,\ \sigma_{22} = \sigma_{11},\ \frac{\partial \sigma_{11}}{\partial x_1} = 0 \ \text{when } -a \leq x_1 \leq 0. \tag{5.2}$$

We similarly formulate the equations of the given approximate solution at $0 \le x_1 \le l/2$. The stress σ_{11} is assumed to be continuous everywhere within the interface.

In accordance with this solution

$$p = (1 - q)\left[2 + \frac{l}{2h}(1 + q)\right], \ m = 0. \tag{5.3}$$

Assuming that a limiting state of compression with shear is realized in one part of the interface and that a limiting state of tension with shear is realized in the other part (with both states being described by equations of the type (5.1)-(5.2) and the stress σ_{11} being continuous everywhere in the layer), we obtain a family of approximate solutions to the problem of the limiting state of the interface. Figure 5a shows the distribution of the normal stresses on the surface $\xi^2 = 1$ corresponding to these solutions. With an assigned length l of interface ($l = l_1 + l_2$), these stresses depend on three parameters: a_1, a_2, and l_1. The dependences of m, p, and q on these parameters can be written in the form

$$q = \frac{a_1 + a_2}{l}, \ p = \frac{l_1 - l_2}{l}\left(2 + \frac{l}{2h}\right) - \frac{a_1 - a_2}{l}\left(2 + \frac{l}{2h}q\right),$$
$$m = 2\alpha(1 - \alpha)\left\{2 - \beta - \gamma + \frac{l}{4h}\left[1 - \alpha\beta^2 - (1 - \alpha)\gamma^2\right]\right\}, \tag{5.4}$$
$$\frac{l_1}{l} = \alpha, \ \frac{l_2}{l} = 1 - \alpha, \ \frac{a_1}{l_1} = \beta, \ \frac{a_2}{l_2} = \gamma.$$

In accordance with (5.4), in the limiting state in bending with shear

$$m = (1 - q)\left[1 + \frac{l}{8h}(1 + q)\right], \ p = 0. \tag{5.5}$$

The dashed lines in Fig. 2b and Fig. 4 show the relation (5.5) between m and q and the corresponding distribution of normal and shear stresses when q = 0.62 and m = 1.42.

In accordance with (5.4), in the limiting state in tension and bending

$$m = \left(1 + \frac{l}{8h}\right)\left[1 - \frac{p^2}{\left(2 + \frac{l}{2h}\right)^2}\right], \ q = 0. \tag{5.6}$$

It follows from the numerical calculations that at $l/h \ge 10$, relation (5.6) between m and p agrees well with the limit loads calculated by the model of interface deformation constructed here.

In the space of m, p, and q, Eqs. (5.4) correspond to a family of surfaces (Fig. 5b). Each of these surfaces passes through curves (5.3), (5.5), (5.6). Taking $a_1/a_2 = l_1/l_2$ in (5.4), we obtain

$$m = \left[1 + \frac{l}{8h}(1 + q)\right]\left\{1 - q - \frac{p^2}{(1 - q)\left[2 + \frac{l}{2h}(1 + q)\right]^2}\right\}. \tag{5.7}$$

Equation (5.7) is one of the simplest possible approximations of the relation between m, p, and q for the limiting state of a straight interface.

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